

# ON REALIZATIONS OF THE GELFAND CHARACTER OF A FINITE GROUP

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**ABSTRACT.** We show that the Gelfand character  $\chi_G$  of a finite group  $G$  (i.e. the sum of all irreducible complex characters of  $G$ ) may be realized as a “twisted trace”  $g \mapsto \text{Tr}(\rho_g \circ T)$  for a suitable involutive linear automorphism of  $L^2(G)$ , where  $\rho$  stands for the right regular representation of  $G$  in  $L^2(G)$ . We prove further that, under certain hypotheses,  $T$  may be obtained as  $T(f) = f \circ L$ , where  $L$  is an involutive antiautomorphism of the group  $G$  so that  $\text{Tr}(\rho_g \circ T) = |\{h \in G : hg = L(h)\}|$ . We also give in the case of the group  $G = PGL(2, \mathbb{F}_q)$  a positive answer to a question of K. W. Johnson asking whether it is possible to express the Gelfand character  $\chi_G$  as a polynomial in a single irreducible character  $\eta$  of  $G$ .

*Keywords:* Gelfand character, twisted trace, total character, Steinberg character, Gelfand Model.

*MSC2010:* 20C15, 20C

## 1. INTRODUCTION

The realization of the Gelfand character  $\chi_G$  of a finite group  $G$ , i.e. the sum of all ordinary irreducible characters of  $G$  is an old problem [6, 3, 9]. One approach to this problem is to try to obtain  $\chi_G$  by twisting the trace of some very natural representation  $(V, \pi)$  of  $G$ , like the regular representation, by a suitable linear automorphism  $T$  of its underlying space  $V$ , so as to obtain  $\chi_G(g) = \text{Tr}(\pi_g \circ T)$  for all  $g \in G$ . Recall that twisted traces appear in many contexts in mathematics [2, 3, 4].

Another possible approach is to try to realize  $\chi_G$  as a polynomial in some remarkable character of  $G$ . In this vein, K. W. Johnson has asked whether it is possible to express the Gelfand character of  $G$  as a polynomial, with integer coefficients, in a single irreducible character  $\eta$  of  $G$ . In [5] an affirmative answer is given for the case of the dihedral groups. In this paper, we consider the case where  $G = PGL(2, q)$ ,  $q$  odd, and we take  $\eta$  to be the Steinberg character of  $G$ . We prove first that the Gelfand character of  $G$  cannot be expressed as a polynomial in the Steinberg character of  $G$  with integer coefficients. We prove then that it can be expressed however as a polynomial, of degree 2, in the Steinberg character of  $G$  with coefficients in

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the ring  $R$  generated over  $\mathbb{Z}$  by the linear characters of  $G$ , i. e. the unit character and the sign character.

## 2. THE GELFAND CHARACTER $\chi_G$ AS A TWISTED TRACE

Let  $G$  be a finite group and let  $(L^2(G), \rho)$  and  $(L^2(G), \sigma)$  be the right and left regular representation of  $G$  respectively; let  $(U_k, \pi_k)$  ( $1 \leq k \leq r$ ) denote all the irreducible unitary representations of  $G$  and  $\mathbf{I}_{\pi^k}$  ( $1 \leq k \leq r$ ) the isotypic component of type  $\pi_k$  of  $\rho$ . Let  $\mathcal{U}_k$  be an orthonormal basis of  $U_k$  and  $(e_{ij}^k(g))_{1 \leq i, j \leq n_k}$  the matrix of the operator  $\pi_k(g)$  ( $g \in G$ ) with respect to the basis  $\mathcal{U}^k$  of  $U^k$ , where  $n_k$  denotes the dimension of  $U_k$ . Moreover  $\chi_k$  denotes the character afforded by  $\pi_k$ .

The matrix coefficients  $e_{ij}^k$  ( $1 \leq i, j \leq n_k, 1 \leq k \leq r$ ) provide then an orthonormal basis  $\mathcal{B}$  for the Hilbert space  $L^2(G)$ , and they satisfy the relations:

$$(1) \quad e_{ij}^k(g^{-1}) = \overline{e_{ji}^k(g)}$$

and

$$e_{ij}^k(gh) = \sum_{l=1}^{n_k} e_{il}^k(g) e_{lj}^k(h)$$

**Proposition 1.** *Let  $(V, \pi^1)$  and  $(V, \pi^2)$  be two isomorphic representations of a finite group  $G$  such that  $\pi_h^1 \circ \pi_g^2 = \pi_g^2 \circ \pi_h^1$  and let  $T$  be an involutive automorphism of  $V$ , that intertwines the representations  $\pi^1$  and  $\pi^2$  of  $G$ . Then the function  $Tr(\pi_g^1 \circ T)$  defined on  $G$  with values on  $\mathbb{C}$  is a central function on  $G$  and so it is a linear complex combination of irreducible characters of  $G$ .*

*Proof.* For  $g, h$  in  $G$ , we have

$$\begin{aligned} Tr(\pi_{g^{-1}hg}^1 \circ T) &= Tr(\pi_g^1 \circ T \circ \pi_{g^{-1}h}^1) = \\ &= Tr(T \circ \pi_g^2 \circ \pi_{g^{-1}h}^1) = Tr(\pi_g^2 \circ T \circ \pi_{g^{-1}h}^1) = \\ &= Tr(\pi_h^2 \circ T) = Tr(T \circ \pi_h^1) = \\ &= Tr(\pi_h^1 \circ T). \end{aligned}$$

□

Recall that all irreducible representations  $\pi^k$  of a finite group  $G$  are unitarizable. The isotypic component  $\mathbf{I}_{\pi^k}$  of type  $(U, \pi^k)$  of  $\rho$  is isomorphic to  $U \otimes U^*$ .

**Theorem 1.** Let  $T$  the linear application of  $L^2(G)$  defined by  $T(e_{ij}^k) = e_{ji}^k$  for all  $e_{ij}^k \in \mathcal{B}$  and the homomorphism  $\tilde{\sigma}$  from  $G$  to  $\text{Aut}(L^2(G))$  defined by

$$\tilde{\sigma}_g(e_{ij}^k) = \sum_{l=1}^{n_k} e_{li}^k(g) e_{lj}^k$$

for all  $e_{ij}^k \in \mathcal{B}$  and for each  $g \in G$ . Then  $T$  is an involutive automorphism of  $L^2(G)$  and  $\tilde{\sigma}$  is a representation of  $G$  such that:

- i.  $\rho_g \circ T = T \circ \tilde{\sigma}_g, \quad g \in G.$
- ii.  $\rho_g \circ \tilde{\sigma}_h = \tilde{\sigma}_h \circ \rho_g, \quad g, h \in G.$
- iii.  $\text{Tr}(\rho_g \circ T) = \chi_G(g), \quad g \in G$

*Proof.* Since  $\tilde{\sigma}_g(e_{ij}^k) = T(\rho_g(T(e_{ij}^k)))$  for all  $g \in G$  and  $e_{ij}^k \in \mathcal{B}$  we obtain that  $\tilde{\sigma}_g$  is an automorphism of  $L^2(G)$  such that  $\rho_g \circ T = T \circ \tilde{\sigma}_g$  for each  $g \in G$ .

Furthermore for  $g, h \in G$  and  $e_{ij}^k \in \mathcal{B}$  verify that

$$\begin{aligned} (\rho_g \circ \tilde{\sigma}_h)(e_{ij}^k) &= \sum_{l=1}^{n_k} e_{li}^k(h) \sum_{m=1}^{n_k} e_{mj}^k(g) e_{lm}^k \\ &= \sum_{m=1}^{n_k} e_{mj}^k(g) \sum_{l=1}^{n_k} e_{li}^k(h) e_{lm}^k \\ &= \sum_{m=1}^{n_k} e_{mj}^k(g) \tilde{\sigma}_h(e_{im}^k) \\ &= (\tilde{\sigma}_h \circ \rho_g)(e_{ij}^k). \end{aligned}$$

Finally, since

$$(\rho_g \circ T)(e_{ij}^k) = \sum_{l=1}^{n_k} e_{li}^k(g) e_{jl}^k$$

for all  $g \in G$  and  $e_{ij}^k \in \mathcal{B}$  then

$$\text{Tr}(\rho_g \circ T) = \sum_{k=1}^r \left( \sum_{i=1}^{n_k} e_{ii}^k(g) \right) = \sum_{k=1}^r \chi^k(g) = \chi_G(g).$$

□

Next we will prove that under certain conditions the central function  $\text{Tr}(\rho_g \circ T)$  can be realized via an involutive anti-automorphism  $L$  of the group  $G$ .

**Theorem 2.** Let  $L$  be an involutive anti-automorphism of  $G$ , such that  $\chi_k(L(g)) = \chi_k(g)$ , for all  $g \in G$  and let  $L^*$  be the automorphism of  $L^2(G)$  defined by  $L^*(f) = f \circ L$ . Then for all  $g \in G$ , we have

- i.  $\rho_g \circ L^* = L^* \circ \sigma_{L(g)^{-1}}$
- ii.  $\rho_g \circ \sigma_{L(g)^{-1}} = \sigma_{L(g)^{-1}} \circ \rho_g$

- iii.  $Tr(\rho_g \circ L^\star) = |\{h \in G : h^{-1}L(h) = g\}|, g \in G$
- iv.  $Tr(\rho_g \circ L^\star) = Tr(\rho_{g^{-1}} \circ L^\star)$
- v.  $Tr(\rho_g \circ L^\star) = \sum_{k=1}^r \varepsilon_k \chi_k(g)$  where  $\varepsilon_k = \pm 1$ .

*Proof.* The proof of i. and ii. is a straightforward calculation. By computing the trace of  $\rho_g \circ L^\star$  with respect to the canonical basis  $\{\delta_g : g \in G\}$  where  $\delta_g(h) = \delta_{g,h}, h \in g$  we obtain iii.

If we notice that  $h^{-1}L(h) = g$  if and only if  $L(h)^{-1}h = g^{-1}$  and  $L$  is one to one we get iv.

Let  $\sigma^\star$  the twisted representation of the left regular representation  $\sigma$  of  $G$  defined by  $\sigma_g^\star = \sigma_{L(g)^{-1}}$ . Due i) and ii) the representations  $(L^2(G), \rho)$  and  $(L^2(G), \sigma^\star)$  of  $G$  satisfy all the conditions of the proposition 1, then we deduce that the complex function  $Tr(\rho_g \circ L^\star)$  is central and

$$Tr(\rho_g \circ L^\star) = \sum_{k=1}^r \varepsilon_k \chi_k(g).$$

The antiautomorphism  $L$  induces an antiautomorphism  $\tilde{L}$  on the complex group algebra  $\mathbb{C}[G]$ . Since  $\chi_k(L(g)) = \chi_k(g), g \in G, 1 \leq k \leq r$ ,  $\tilde{L}$  acts as the identity on the center and therefore induces an antiautomorphism  $\tilde{L}_k$  on each simple component of  $\mathbb{C}[G] \cong \bigoplus_{1 \leq k \leq r} M(n_k, \mathbb{C})$ . Due to Skolem-Noether theorem,  $\tilde{L}_k$  is conjugated to the transposition:  $\tilde{L}_k(a) = b^{-1}a^t b, a \in M(n_k, \mathbb{C})$  and  $b \in Gl(n_k, \mathbb{C})$ . Furthermore we have that  $a = \tilde{L}(\tilde{L}(a)) = b^{-1}b^t a b^{-1}b^t$ , then  $b^{-1}b^t$  belongs to the center and therefore  $b^t = \varepsilon_k b$  with  $\varepsilon_k = \pm 1$ .

In this way, for each representation  $(U_k, \pi_k)$  of  $G$  a symmetric form or a symplectic form  $b$  exists, with respect to which the linear operators  $\pi_k(g)$  and  $\tilde{L}(\pi_k(g))$  are conjugated. Therefore if we consider the bilinear form  $\langle u, v \rangle = v^t b u$  then

$$(2) \quad \langle \tilde{L}(\pi_k(g))(u), v \rangle = v^t (b \tilde{L}(\pi_k(g))) u = v^t (\pi_k(g))^t b u = \langle u, \pi_k(g)(v) \rangle$$

Let us suppose that  $\varepsilon_k = 1$  for some  $k$ . We choose an orthonormal basis  $\mathcal{U}_k^+ = \{u_i, 1 \leq i \leq n_k\}$  of  $U_k$ , respect to the symmetric form  $b$  and we denote by  $e_{ij}^k(g) = \langle \pi_k(g)(u_j), u_i \rangle$  the matrix coefficients of  $\pi_k(g)$  with respect to this basis. Due to eq.2 we have the following relations between the matrix coefficients of  $\tilde{L}(\pi_k(g))$  and  $\pi_k(g)$

$$(3) \quad e_{ij}^k(L(g)) = \langle u_j, \pi_k(g)(u_i) \rangle = e_{ji}^k(g).$$

Let  $E^k = \langle e_{ij}^k(g) : 1 \leq i, j \leq n_k \rangle$  and  $Tr_k(\rho_g \circ L^\star)$  the restriction of  $Tr(\rho_g \circ L^\star)$  to the subspace  $E^k$ . In order to compute  $Tr_k(\rho_g \circ L^\star)$  we note that

$$(\rho_g \circ L^\star)(e_{ij}^k)(h) = e_{ij}^k(L(hg)) = e_{ij}^k(L(g)L(h)) =$$

$$\sum_{l=1}^{n_k} e_{li}^k(L(g))e_{lj}^k(L(h)) = \sum_{l=1}^{n_k} e_{li}^k(g)e_{jl}^k(h).$$

Then

$$(\rho_g \circ L^*)(e_{ij}^k) = \sum_{l=1}^{n_k} e_{li}^k(g)e_{jl}^k.$$

Since

$$Tr_k(\rho_g \circ L^*) = \sum_{1 \leq i, j \leq n_k} \langle (\rho_g \circ L^*)(e_{ij}^k), e_{ij}^k \rangle = \sum_{1 \leq i, j \leq n_k} \sum_{l=1}^{n_k} e_{li}^k(g) \langle e_{jl}^k(h), e_{ij}^k \rangle$$

and  $\langle e_{jl}^k, e_{ij}^k \rangle$  is equal to 1 when  $j = i, l = j$  and is equal to 0 otherwise, we get

$$Tr_k(\rho_g \circ L^*) = \sum_{i=1}^{n_k} e_{ii}^k(g) = \chi_k(g)$$

Let us suppose that  $\varepsilon_k = -1$  for some  $k$ . Let  $n = \frac{n_k}{2}$ , then we can find a basis  $\mathcal{U}_k^- = \{u_i, 1 \leq i \leq n_k\}$  of  $U_k$  such that  $\langle u_i, u_{i+n} \rangle = 1$  and  $\langle u_i, u_{j+n} \rangle = \langle u_{i+n}, u_{j+n} \rangle = \langle u_i, u_j \rangle = 0, i \neq j, i, j = 1, \dots, n_k$

Analogously, equation 1 gives us the following relations for the matrix coefficients  $e_{ij}^k(g)$  of  $\pi_k(g)$  with respect to this basis:

$$\begin{aligned} \langle \pi_k(g)(u_i), u_j \rangle &= -e_{j+ni}^k(g), j \leq n, \\ \langle \pi_k(g)(u_i), u_j \rangle &= e_{j-ni}^k(g), j > n \end{aligned}$$

Therefore

$$\langle \pi_k(L(g))(u_i), u_j \rangle = -\langle \pi_k(g)(u_j), u_i \rangle = e_{i+nj}^k(g), i \leq n,$$

and

$$\langle \pi_k(L(g))(u_i), u_j \rangle = -e_{i-nj}^k(g), i > n$$

Taking account these relations and equation 2 we get the following relations between the matrix coefficients of  $\tilde{L}(\pi_k(g))$  and  $\pi_k(g)$

$$(4) \quad e_{ij}^k(L(g)) = e_{j+ni-n}^k(g), j \leq n, i > n$$

$$(5) \quad e_{ij}^k(L(g)) = -e_{j-ni-n}^k(g), j > n, i > n$$

$$(6) \quad e_{ij}^k(L(g)) = -e_{j+ni+n}^k(g), j \leq n, i \leq n$$

$$(7) \quad e_{ij}^k(L(g)) = e_{j-ni+n}^k(g), j > n, i \leq n$$

In this case we get: For  $i, j \leq n$

$$(8) \quad (\rho_g \circ L^*)(e_{ij}^k) = \sum_{l=1}^n -e_{l+ni+n}^k(g)e_{j+nl+n}^k + \sum_{l=n+1}^{n_k} e_{l-ni+n}^k(g)e_{j+nl-n}^k.$$

For  $i \leq n, j > n$

$$(9) \quad (\rho_g \circ L^*)(e_{ij}^k) = \sum_{l=1}^n -e_{l+ni+n}^k(g)e_{j-nl+n}^k + \sum_{l=n+1}^{n_k} e_{l-ni+n}^k(g)(-e_{j-nl-n}^k).$$

For  $i > n, j \leq n$

$$(10) \quad (\rho_g \circ L^*)(e_{ij}^k) = \sum_{l=1}^n e_{l+ni-n}^k(g)(-e_{j+nl+n}^k) + \sum_{l=n+1}^{n_k} -e_{l-ni-n}^k(g)e_{j+nl-n}^k.$$

And for  $i > n, j > n$

$$(11) \quad (\rho_g \circ L^*)(e_{ij}^k) = \sum_{l=1}^n e_{l+ni-n}^k(g)e_{j-nl+n}^k + \sum_{l=n+1}^{n_k} -e_{l-ni-n}^k(g)(-e_{j-nl-n}^k).$$

Notice that:

- a) If  $i, j \leq n$  then  $e_{j+nl+n}^k \neq e_{ij}^k$  and  $e_{j+nl-n}^k \neq e_{ij}^k$
- b) If  $i \leq n, j > n$  then  $e_{j-nl+n}^k = e_{ij}^k$  if and only if  $j = n + i > n$  and  $l + n = j > n$  and  $e_{j-nl-n}^k \neq e_{ij}^k$
- c) If  $i > n, j \leq n$  then  $e_{j+nl-n}^k = e_{ij}^k$  for  $j + n = i > n$  and  $l - n = j < n$  and  $e_{j+nl+n}^k \neq e_{ij}^k$
- d) If  $i > n, j > n$  then  $e_{j-nl+n}^k \neq e_{ij}^k$  and  $e_{j+nl-n}^k \neq e_{ij}^k$

Therefore

$$Tr_k(\rho_g \circ L^*) = \sum_{j=n+1}^{n_k} -e_{jj}^k(g) + \sum_{j=1}^n -e_{jj}^k(g) = -\chi_k(g)$$

so that

$$Tr(\rho_g \circ L^*) = \sum_{k=1}^r Tr_k(\rho_g \circ L^*) = \sum_{k=1}^r \varepsilon_k \chi_k(g)$$

□

**Proposition 2.** *If  $L$  is an involutive antiautomorphism of  $G$  such that :*

$$\chi_k(L(g)) = \chi_k(g),$$

*for  $1 \leq k \leq r$  and*

$$|\{g \in G : L(g) = g\}| = \sum_{k=1}^r n_k.$$

then

$$\text{Tr}(\rho_g \circ L^\star) = \sum_{k=1}^r \chi_k(g)$$

and

$$L^\star = T.$$

*Proof.* If we evaluate  $\text{Tr}(\rho_g \circ L^\star)$  on  $e$  we obtain

$$\text{Tr}(L^\star) = \sum_{k=1}^r \varepsilon_k n_k$$

but  $\text{Tr}(L^\star) = |\{g \in G : L(g) = g\}|$  and by hypothesis

$$|\{g \in G : L(g) = g\}| = \sum_{k=1}^r n_k.$$

Since  $n_k > 0$ , we conclude that  $\varepsilon_k = 1$  for all  $k$ . Then using eq.3 it follows that

$$L^\star(e_{ij}^k)(g) = e_{ij}^k(L(g)) = e_{ji}^k(g)$$

ie  $L^\star(e_{ij}^k) = e_{ji}^k$ .

□

**Proposition 3.** *If  $L$  is an involutive antiautomorphism of  $G$  such that :*

$$\chi_k(L(g)) = \chi_k(g),$$

for  $1 \leq k \leq r$  then

$$(12) \quad \text{Tr}_k(\rho_g \circ L^\star) = \left( \frac{1}{|G|} \sum_{h \in G} \chi_k(L(h)h^{-1}) \right) \chi_k(g)$$

*Proof.* To prove equation (12) we compute the Fourier coefficients  $\lambda_k, 1 \leq k \leq r$  of the central function

$$\text{Tr}(\rho_g \circ L^\star) = \text{Tr}(L^\star \circ \sigma_{L(g)^{-1}})$$

with respect to the basis

$$\{\chi_k = \sum_{i=1}^{n_k} e_{ii}^k : 1 \leq k \leq r\}$$

of  $\mathbb{Z}[G]$ .

First we observe that

$$(L^\star \circ \sigma_{L(g)^{-1}})(e_{ij}^k)(h) = e_{ij}^k(L(g)L(h)) = \sum_{l=1}^{n_k} e_{il}^k(L(g))(e_{lj}^k \circ L)(h)$$

and

$$\langle (L^\star \circ \sigma_{L(g)^{-1}})(e_{ij}^k), e_{ij}^k \rangle = \frac{1}{|G|} \sum_{h \in G} \sum_{l=1}^{n_k} e_{il}^k(L(g)) e_{il}^k(L(h)) e_{ji}^k(h^{-1}).$$

therefore

$$Tr(L^\star \circ \sigma_{L(g)^{-1}}) = \sum_{k=1}^r \sum_{1 \leq i, j \leq n_k} \frac{1}{|G|} \sum_{h \in G} \sum_{l=1}^{n_k} e_{il}^k(L(g)) e_{lj}^k(L(h)) e_{ji}^k(h^{-1})$$

Since

$$\sum_{j=1}^{n_k} e_{lj}^k(L(h)) e_{ji}^k(h^{-1}) = e_{li}^k(L(h)) h^{-1}$$

we get that

$$Tr(L^\star \circ \sigma_{L(g)^{-1}}) = \sum_{k=1}^r \sum_{1 \leq i, l \leq n_k} \frac{1}{|G|} \sum_{h \in G} e_{li}^k(L(h) h^{-1}) (e_{il}^k \circ L)(g)$$

and then

$$\lambda_{k'} = \left\langle \sum_{k=1}^r \sum_{1 \leq i, l \leq n_k} \frac{1}{|G|} \sum_{h \in G} e_{li}^k(L(h) h^{-1}) (e_{il}^k \circ L), \chi_{k'} \right\rangle$$

By hypothesis  $\chi_k(L(g)) = \chi_k(g)$ ,  $1 \leq k \leq r$ , then

$$\lambda_{k'} = \sum_{k=1}^r \sum_{1 \leq i, l \leq n_k} \frac{1}{|G|} \sum_{h \in G} e_{li}^k(L(h) h^{-1}) \sum_{s=1}^{n_{k'}} \left( \frac{1}{|G|} \sum_{g \in G} e_{il}^k(L(g)) \overline{e_{ss}^{k'}(L(g))} \right),$$

But  $\frac{1}{|G|} \sum_{g \in G} e_{il}^k(L(g)) \overline{e_{ss}^{k'}(L(g))} = \langle e_{il}^k, e_{ss}^{k'} \rangle$ , then

$$\lambda_{k'} = \sum_{s=1}^{n_{k'}} \frac{1}{|G|} \sum_{h \in G} e_{ss}^{k'}(L(h) h^{-1}) = \frac{1}{|G|} \sum_{h \in G} \chi_{k'}(L(h) h^{-1}),$$

□

**Proposition 4.** *If  $L$  be an involutive antiautomorphism of  $G$  such that*

$$\chi_k(L(g)) = \chi_k(g)$$

*for all  $1 \leq k \leq r$  then the following conditions are equivalent*

i.

$$|\{g \in G : L(g) = g\}| = \sum_{k=1}^r n_k$$

ii.

$$\frac{1}{|G|} \sum_{h \in G} \chi_k(L(h) h^{-1}) = 1, \quad (1 \leq k \leq r)$$



*Proof.* This follows from propositions 2 and 3 □

**Proposition 5.** *Let  $L$  an involutive antiautomorphism of  $G$ ,  $\tau : G \rightarrow G$  defined by  $\tau(g) = L(g^{-1})$  and  $c_\tau(\chi) = |G|^{-1} \sum_{g \in G} \chi(g\tau(g))$  the twisted Frobenius - Schur indicator defined by Kawanaka and Matsuyama. Then  $c_\tau(\chi_k) = \frac{1}{|G|} \sum_{h \in G} \chi_k(hL(h^{-1})) = \pm 1$  if  $\chi_k(L(g)) = \chi_k(g)$ . And if the matrix representations  $R_k$  afforded by  $\chi_k$ , satisfies  $R_k(L(g)) = R_k(g)$ , then  $c_\tau(\chi_k) = 1$ .*

*Proof.* Due to  $g\tau(g) = gL(g^{-1})$ , we have

$$\sum_{g \in G} \chi(gL(g^{-1})) = \sum_{g \in G} \overline{\chi(L(g)g^{-1})}$$

and since

$$Tr(\rho_g \circ L^*) = \overline{Tr(\rho_g \circ L^*)}$$

we can write:

$$Tr(\rho_g^{-1} \circ L^*) = \sum_{k=1}^r \frac{1}{|G|} \sum_{h \in G} \overline{\chi_k(L(h)h^{-1})} \chi_k(g^{-1}),$$

and therefore

$$Tr(\rho_g^{-1} \circ L^*) = \sum_{k=1}^r \frac{1}{|G|} \sum_{h \in G} \chi_k(hL(h^{-1})) \chi_k(g).$$

This proves that  $\frac{1}{|G|} \sum_{h \in G} \chi_k(hL(h^{-1})) = \pm 1$  if  $\chi_k(L(g^{-1})) = \overline{\chi_k(g)}$  ie  $\chi_k(L(g)) = \chi_k(g)$ , and if the matrix representations  $R_k$  afforded by  $\chi_k$ , satisfies  $R_k(L(g)) = R_k(g)$ , then  $c_\tau(\chi_k) = 1$ . □

### 3. THE GEL'FAND CHARACTER AS A POLYNOMIAL IN THE STEINBERG CHARACTER

We consider below the case of the group  $G = PGL(2, k)$ ,  $k$  a finite field of any characteristic. We denote by  $T$  the Coxeter torus of  $G$ .

**Proposition 6.** *The Gelfand character  $\chi_G$  of  $G$  may be expressed as*

$$\chi_G = St^2 + St + sgn \cdot \mathbf{1}$$

*as a polynomial in  $St$  with coefficients in the ring  $R = \mathbb{Z}[\mathbf{1}, sgn]$  generated by the one dimensional characters of  $G$ .*

*Proof.* In [1] it was proved that

$$St^2 = Ind_T^G \mathbf{1} + St$$

a fact that can be checked by a straightforward character calculation.

Now the theorem follows from the explicit decomposition of  $Ind_T^G \mathbf{1}$  obtained in [7].

□

**Corollary 1.** *Neither the sign character  $sgn$  nor the Gelfand character of  $G$  belong to the ring  $\mathbb{Z}[St]$ .*

Indeed, just notice that the Steinberg character and all its powers are constant on hyperbolic elements (value 1) and on elliptic elements (value - 1). There are however hyperbolic as well as elliptic elements whose sign (class of the determinant modulo squares) is +1 as well as -1. So it is impossible that the sign character  $sgn$  be a polynomial in the Steinberg character.

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